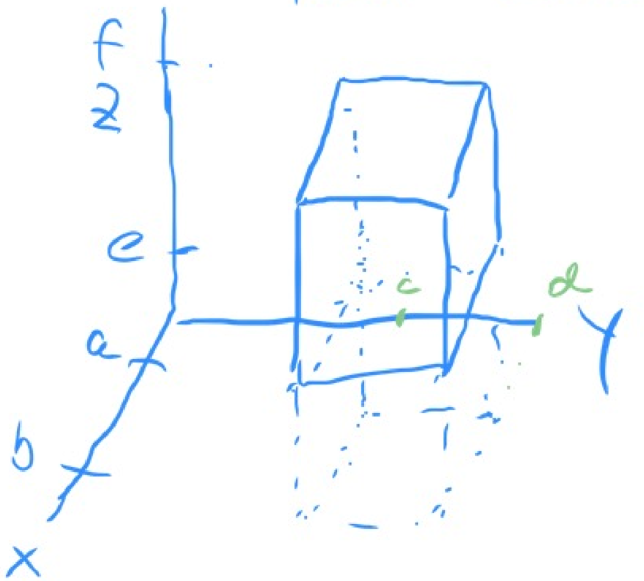
Triple Integrals

$$f: W \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$$

integrate $\iiint_W f \, dV$

simplest case: $W = $ a box



$$= [a,b] \times [c,d] \times [e,f]$$

$$= \{ (x,y,z), \; \begin{array}{l} a \leq x \leq b \\ c \leq y \leq d \\ e \leq z \leq f \end{array} \}$$

Example: $W =$ box $[0,1] \times [1,2] \times [0,2]$

Calculate $\iiint_W xyz \, dV$

$$= \int_0^1 \int_1^2 \int_0^2 xyz \, dz \, dy \, dx$$

$$= \int_0^1 \int_1^2 xy \frac{z^2}{2} \Big|_{z=0}^{z=2} dy \, dx$$

$$= \int_0^1 \int_1^2 xy2 - 0 \; dy \, dx$$

$$= \int_0^1 xy^2 \Big|_{y=1}^{y=2} dx = \int_0^1 3x \, dx = \frac{3}{2}$$

generalize concept of y-simple region (elementary region) to 3 dimensions:

<u>Def.</u> A region $W \subset \mathbb{R}^3$ is called elementary if it can be described as set of points (x,y,z) satisfying

$$a \leq x \leq b$$

for fixed x $\quad \phi_1(x) \leq y \leq \phi_2(x)$ $\quad$ for two functions $\phi_1(x) \leq \phi_2(x)$

for fixed x,y $\quad \gamma_1(x,y) \leq z \leq \gamma_2(x,y)$ $\quad$ " " " $\gamma_1(x,y) \leq \gamma_2(x,y)$

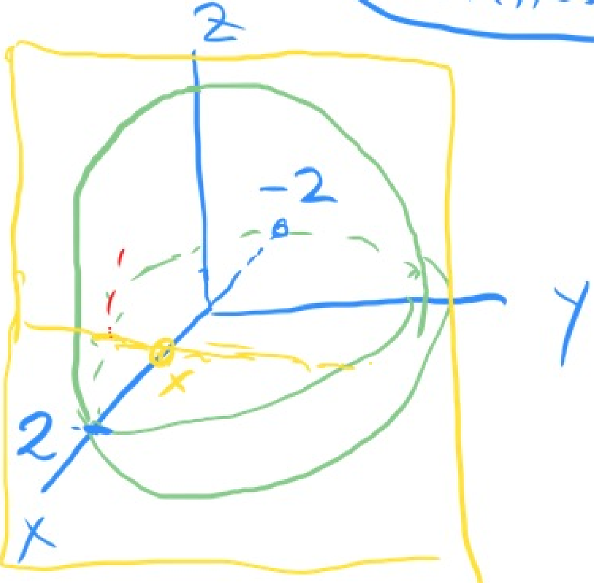
Examples: (1) Describe ball B of radius 2 as an elementary region

aside: definition of elementary region also works for permutations of coordinate

e.g. $a \le z \le b$

$\phi_1(z) \le y \le \phi_2(z)$

$\gamma_1(y,z) \le x \le \gamma_2(y,z)$



B given by equation $\boxed{x^2+y^2+z^2 \le 2^2}$

possible x-values: $\boxed{-2 \le x \le 2}$

fix x: poss. y-values

$$y^2 + \underbrace{z^2}_{\ge 0} \le \underbrace{4-x^2}_{\text{fixed}}$$

$$\Rightarrow y^2 \le 4 - x^2$$

$$\Rightarrow \boxed{-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}}$$

describes all possible y values of intersection with yellow plane

fix x, y: $\Rightarrow z^2 \le 4 - x^2 - y^2 \Rightarrow \boxed{-\sqrt{4-x^2-y^2} \le z \le \sqrt{4-x^2-y^2}}$

<u>Remark:</u> · Determine all possible x-values

- fix an x-value, say x_0

$\Longrightarrow$ get 2-dim region D from intersection of W with plane $x = x_0$.

- describe (y,z)-coordinates of D as an elementary region in y and z.

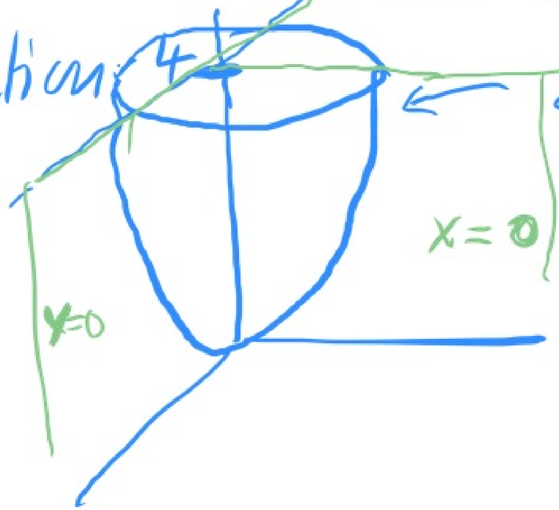
in our example: intersection $= \{(x_0, y, z),\ x_0^2 + y^2 + z^2 \leq 4\}$

$\Longrightarrow$ (y,z) coordinates describe disk given by $y^2 + z^2 \leq 4 - x_0^2$

usually we just write x for x_0

Example 2: Describe the region W bounded by planes $x=0$, $y=0$, $z=4$ and above $z=x^2+y^2$ as an elementary region

Solution:



$z=x^2+y^2$ paraboloid.

$z\le 4$

additional conditions:

$x \ge 0$

$y \ge 0$

get one quarter of paraboloid $z=x^2+y^2$ with $z\le 4$

$y=0 \iff xz$ plane

$x=0 \iff yz$ plane

we got inequalities $x \geq 0$

$y \geq 0$

$x^2+y^2 \leq z \leq 4$

possible x-values: $x \geq 0$ and $x^2+y^2 \leq 4$

$\Rightarrow x^2 \leq 4$

$\Rightarrow \boxed{0 \leq x \leq 2}$

Fix ~~x~~ x:

$\boxed{0 \leq y}$

$x^2+y^2 \leq 4$

$\Rightarrow y^2 \leq 4-x^2$

$\boxed{y \leq \sqrt{4-x^2}}$

$\Rightarrow \boxed{0 \leq y \leq \sqrt{4-x^2}}$

Fix x and y: $\Rightarrow \boxed{x^2+y^2 \leq z \leq 4}$